18.175 Theory of Probability
Fall 2008

Section 13

Martingales. Doob's Decomposition. Uniform Integrability.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and let $(T, \leq)$ be a linearly ordered set. Consider a family of σ-algebras $\mathcal{B}_t, t \in T$ such that for $t \leq u$, $\mathcal{B}_t \subseteq \mathcal{B}_u \subseteq \mathcal{B}$.

Definition. A family $(X_t, \mathcal{B}_t)_{t \in T}$ is called a *martingale* if

1. $X_t : \Omega \to \mathbb{R}$ is measurable w.r.t. $\mathcal{B}_t$; in other words, X_t is *adapted* to $\mathcal{B}_t$.
2. $\mathbb{E}|X_t| < \infty$.
3. $\mathbb{E}(X_u|\mathcal{B}_t) = X_t$ for $t \leq u$.

If the last equality is replaced by $\mathbb{E}(X_u|\mathcal{B}_t) \leq X_t$ then the process is called a *supermartingale* and if $\mathbb{E}(X_u|\mathcal{B}_t) \geq X_t$ then it is called a *submartingale.*

Examples.

1. Consider a sequence $(X_n)_{n \geq 1}$ of independent random variables such that $\mathbb{E}X_i = 0$ and let $S_n = \sum_{i \leq n} X_i$. If $\mathcal{B}_n = \sigma(X_1, \ldots, X_n)$ is a σ-algebra generated by the first n r.v.s then $(S_n, \mathcal{B}_n)_{n \geq 1}$ is a martingale since

$$\mathbb{E}(S_{n+1}|\mathcal{B}_n) = \mathbb{E}(X_{n+1} + S_n|\mathcal{B}_n) = 0 + S_n = S_n.$$

2. Consider a sequence of σ-algebras

$$\ldots \subseteq \mathcal{B}_m \subseteq \mathcal{B}_n \subseteq \ldots \subseteq \mathcal{B}$$

and a r.v. X on $\mathcal{B}$ and let $X_n = \mathbb{E}(X|\mathcal{B}_n)$. Then $(X_n, \mathcal{B}_n)$ is a martingale since for $m < n$

$$\mathbb{E}(X_n|\mathcal{B}_m) = \mathbb{E}(\mathbb{E}(X|\mathcal{B}_n)|\mathcal{B}_m) = \mathbb{E}(X|\mathcal{B}_m) = X_m.$$

Definition. If $(X_n, \mathcal{B}_n)$ is a martingale and for some r.v. X, $X_n = \mathbb{E}(X|\mathcal{B}_n)$, then the martingale is called *right-closable.* If $X_\infty = X, \mathcal{B}_\infty = \mathcal{B}$ then $(X_n, \mathcal{B}_n)_{n \leq \infty}$ is called *right-closed.*

3. Let $(X_i)_{i \geq 1}$ be i.i.d. and let $S_n = \sum_{i \leq n} X_i$. Let us take $T = \{\ldots, -2, -1\}$ and for $n \geq 1$ define

$$\mathcal{B}_{-n} = \sigma(S_n, S_{n+1}, \ldots) = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots).$$

Clearly, $\mathcal{B}_{-(n+1)} \subseteq \mathcal{B}_{-n}$. For $1 \leq k \leq n$, by symmetry,

$$\mathbb{E}(X_1|\mathcal{B}_{-n}) = \mathbb{E}(X_k|\mathcal{B}_{-n}).$$

Therefore,

$$S_n = \mathbb{E}(S_n|\mathcal{B}_{-n}) = \sum_{1 \leq k \leq n} \mathbb{E}(X_k|\mathcal{B}_{-n}) = n\mathbb{E}(X_1|\mathcal{B}_{-n}) \Longrightarrow Z_{-n} := \frac{S_n}{n} = \mathbb{E}(X_1|\mathcal{B}_{-n}).$$

Thus, $(Z_{-n}, \mathcal{B}_{-n})_{-n\le -1}$ is a right-closed martingale.

$\square$

Lemma 28 *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Suppose that either one of two conditions holds:*

1. *$(X_t, \mathcal{B}_t)$ is a martingale,*
2. *$(X_t, \mathcal{B}_t)$ is a submartingale and f is increasing.*

Then $(f(X_t), \mathcal{B}_t)$ is a submartingale.

Proof. 1. For $t \le u$, by Jensen's inequality,

$$f(X_t) = f(\mathbb{E}(X_u|\mathcal{B}_t)) \le \mathbb{E}(f(X_u)|\mathcal{B}_t).$$

2. For $t \le u$, since $X_t \le \mathbb{E}(X_u|\mathcal{B}_t)$ and f is increasing,

$$f(X_t) \le f(\mathbb{E}(X_u|\mathcal{B}_t)) \le \mathbb{E}(f(X_u)|\mathcal{B}_t),$$

where the last step is again Jensen's inequality.

$\square$

Theorem 30 *(Doob's decomposition) If $(X_n, \mathcal{B}_n)_{n\ge 0}$ is a submartingale then it can be uniquely decomposed*

$$X_n = Z_n + Y_n,$$

where $(Y_n, \mathcal{B}_n)$ is a martingale, $Z_0 = 0, Z_n \le Z_{n+1}$ almost surely and Z_n is $\mathcal{B}_{n-1}$-measurable.

Proof. Let $D_n = X_n - X_{n-1}$ and

$$G_n = \mathbb{E}(D_n|\mathcal{B}_{n-1}) = \mathbb{E}(X_n|\mathcal{B}_{n-1}) - X_{n-1} \ge 0$$

by the definition of submartingale. Let,

$$H_n = D_n - G_n, \;\; Y_n = H_1 + \ldots + H_n, \;\; Z_n = G_1 + \cdots + G_n.$$

Since $G_n \ge 0$ a.s., $Z_n \le Z_{n+1}$ and, by construction, Z_n is $\mathcal{B}_{n-1}$-measurable. We have,

$$\mathbb{E}(H_n|\mathcal{B}_{n-1}) = \mathbb{E}(D_n|\mathcal{B}_{n-1}) - G_n = 0$$

and, therefore, $\mathbb{E}(Y_n|\mathcal{B}_{n-1}) = Y_{n-1}$. Uniqueness follows by construction. Suppose that $X_n = Z_n + Y_n$ with all stated properties. First, since $Z_0 = 0$, $Y_0 = X_0$. By induction, given a unique decomposition up to $n-1$, we can write

$$Z_n = \mathbb{E}(Z_n|\mathcal{B}_{n-1}) = \mathbb{E}(X_n - Y_n|\mathcal{B}_{n-1}) = \mathbb{E}(X_n|\mathcal{B}_{n-1}) - Y_{n-1}$$

and $Y_n = X_n - Z_n$.

$\square$

Definition. We say that $(X_n)_{n\ge 1}$ is uniformly integrable if

$$\sup_n \mathbb{E}|X_n| < \infty \quad \text{and} \quad \sup_n \mathbb{E}|X_n| \mathrm{I}(|X_n| > M) \to 0 \quad \text{as} \quad M \to \infty.$$

Lemma 29 *The following holds.*

1. *If $(X_n, \mathcal{B}_n)$ is a right-closable martingale then (X_n) is uniformly integrable.*
2. *If $(X_n, \mathcal{B}_n)_{n\le\infty}$ is a submartingale then for any $a \in \mathbb{R}$, $(\max(X_n, a))$ is uniformly integrable.*

Proof. 1. If $X_n = \mathbb{E}(Y|\mathcal{B}_n)$ then

$$|X_n| = |\mathbb{E}(Y|\mathcal{B}_n)| \le \mathbb{E}(|Y||\mathcal{B}_n) \quad \text{and} \quad \mathbb{E}|X_n| \le \mathbb{E}|Y| < \infty.$$

Since $\{|X_n| > M\} \in \mathcal{B}_n$,

$$X_n \mathrm{I}(|X_n| > M) = \mathrm{I}(|X_n| > M)\mathbb{E}(Y|\mathcal{B}_n) = \mathbb{E}(Y\mathrm{I}(|X_n| > M)|\mathcal{B}_n)$$

and, therefore,

$$\begin{aligned}\mathbb{E}|X_n|\mathrm{I}(|X_n| > M) \le \mathbb{E}|Y|\mathrm{I}(|X_n| > M) &\le K\mathbb{P}(|X_n| > M) + \mathbb{E}|Y|\mathrm{I}(|Y| > K)\\ &\le K\frac{\mathbb{E}|X_n|}{M} + \mathbb{E}|Y|\mathrm{I}(|Y| > K) \le K\frac{\mathbb{E}|Y|}{M} + \mathbb{E}|Y|\mathrm{I}(|Y| > K).\end{aligned}$$

Letting $M \to \infty, K \to \infty$ proves that $\sup_n \mathbb{E}|X_n|\mathrm{I}(|X_n| > M) \to 0$ as $M \to \infty$.

2. Since $(X_n, \mathcal{B}_n)_{n\le\infty}$ is a submartingale, for $Y = X_\infty$ we have $X_n \le \mathbb{E}(Y|\mathcal{B}_n)$. Below we will use the following observation. Since a function $\max(a, x)$ is convex and increasing in x, by Jensen's inequality

$$\max(a, X_n) \le \mathbb{E}(\max(a, Y)|\mathcal{B}_n). \tag{13.0.1}$$

Since,

$$\big|\max(X_n, a)\big| \le |a| + X_n\mathrm{I}\big(X_n > |a|\big)$$

and $\{|X_n| > |a|\} \in \mathcal{B}_n$ we can write

$$\mathbb{E}\big|\max(X_n, a)\big| \le |a| + \mathbb{E}X_n\mathrm{I}\big(X_n > |a|\big) \le |a| + \mathbb{E}Y\mathrm{I}\big(X_n > |a|\big) \le |a| + \mathbb{E}|Y| < \infty.$$

If we take $M > |a|$ then

$$\begin{aligned}\mathbb{E}|\max(X_n, a)|\mathrm{I}(|\max(X_n, a)| > M) &= \mathbb{E}X_n\mathrm{I}(X_n > M) \le \mathbb{E}Y\mathrm{I}(X_n > M)\\ &\le K\mathbb{P}(X_n > M) + \mathbb{E}|Y|\mathrm{I}(|Y| > K)\\ &\le K\frac{\mathbb{E}\max(X_n, 0)}{M} + \mathbb{E}|Y|\mathrm{I}(|Y| > K)\\ \text{by (13.0.1)} \quad &\le K\frac{\mathbb{E}\max(Y, 0)}{M} + \mathbb{E}|Y|\mathrm{I}(|Y| > K).\end{aligned}$$

Letting $M \to \infty$ and $K \to \infty$ finishes the proof.

□

Uniform integrability plays an important role when studying the convergence of martingales. The following strengthening of the dominated convergence theorem will be useful.

Lemma 30 *Consider r.v.s (X_n) and X such that $\mathbb{E}|X_n| < \infty, \mathbb{E}|X| < \infty$. Then the following are equivalent:*

1. $\mathbb{E}|X_n - X| \to 0$,
2. *(X_n) is uniformly integrable and $X_n \to X$ in probability.*

Proof. 2$\Longrightarrow$1. We can write,

$$\begin{aligned}\mathbb{E}|X_n - X| &\le \varepsilon + \mathbb{E}|X_n - X|\mathrm{I}(|X_n - X| > \varepsilon)\\ &\le \varepsilon + 2K\mathbb{P}(|X_n - X| > \varepsilon) + 2\mathbb{E}|X_n|\mathrm{I}(|X_n| > K) + 2\mathbb{E}|X|\mathrm{I}(|X| > K)\\ &\le \varepsilon + 2K\mathbb{P}(|X_n - X| > \varepsilon) + 2\sup_n \mathbb{E}|X_n|\mathrm{I}(|X_n| > K) + 2\mathbb{E}|X|\mathrm{I}(|X| > K).\end{aligned}$$

Letting $n \to \infty$ and then $\varepsilon \to 0, K \to \infty$ proves the result.

1$\Longrightarrow$2. By Chebyshev's inequality,

$$\mathbb{P}(|X_n - X| > \varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}|X_n - X| \to 0$$

as $n \to \infty$ so $X_n \to X$ in probability. To prove uniform integrability let us first show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbb{P}(A) < \delta \Longrightarrow \mathbb{E}|X|\mathrm{I}_A < \varepsilon.$$

Suppose not. Then, for some $\varepsilon > 0$ one can find a sequence of events $A(n)$ such that

$$\mathbb{P}(A(n)) \le \frac{1}{2^n} \quad \text{and} \quad \mathbb{E}|X|\mathrm{I}_{A(n)} > \varepsilon.$$

Since $\sum_{n\ge 1} \mathbb{P}(A(n)) < \infty$, by Borel-Cantelli lemma, $\mathbb{P}(A(n) \text{ i.o.}) = 0$. This means that $|X|\mathrm{I}_{A(n)} \to 0$ almost surely and by the dominated convergence theorem $\mathbb{E}|X|\mathrm{I}_{A(n)} \to 0$ - a contradiction.

Given $\varepsilon > 0$, take δ as above and take $M > 0$ large enough so that for all $n \ge 1$

$$\mathbb{P}(|X_n| > M) \le \frac{\mathbb{E}|X_n|}{M} < \delta.$$

Then,

$$\mathbb{E}|X_n|\mathrm{I}(|X_n| > M) \le \mathbb{E}|X_n - X| + \mathbb{E}|X|\mathrm{I}(|X_n| > M) \le \mathbb{E}|X_n - X| + \varepsilon.$$

For large enough $n \ge n_0$, $\mathbb{E}|X_n - X| \le \varepsilon$ and, therefore,

$$\mathbb{E}|X_n|\mathrm{I}(|X_n| > M) \le 2\varepsilon.$$

We can also choose M large enough so that $\mathbb{E}|X_n|\mathrm{I}(|X_n| > M) \le 2\varepsilon$ for $n \le n_0$ and this finishes the proof.

$\square$